A few review problems for 1.7 & 1.8

Exercises and Problems for Section 1.7

EXERCISES

1. (a) Using Figure 1.108, find all values of x for which f is not continuous.
List the largest open intervals on which f is continuous.

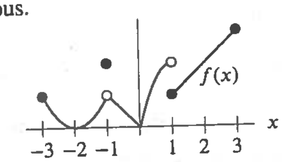

Figure 1.108

Using Figure 1.109, find all values of x for which f is not continuous.
List the largest open intervals on which f is continuous.

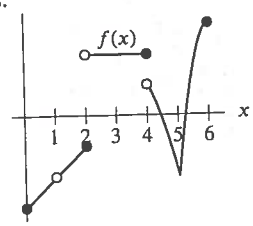

Figure 1.109

3. Use the graph of $f(x)$ in Figure 1.110 to give approximate values for the following limits (if they exist). If the limit does not exist, say so.

(a) $\lim_{x\to -3} f(x)$ (b) $\lim_{x\to -2} f(x)$ (c) $\lim_{x\to -1} f(x)$
(d) $\lim_{x\to 0} f(x)$ (e) $\lim_{x\to 1} f(x)$ (f) $\lim_{x\to 3} f(x)$

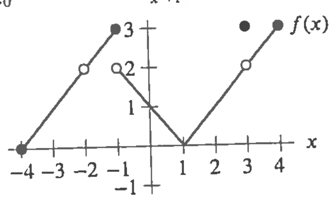

Figure 1.110

In Exercises 4–5, the graph of $y = f(x)$ is given.
(a) Give the x-values where $f(x)$ is not continuous.
(b) Does the limit of $f(x)$ exist at each x-value where $f(x)$ is not continuous? If so, give the value of the limit.

4.

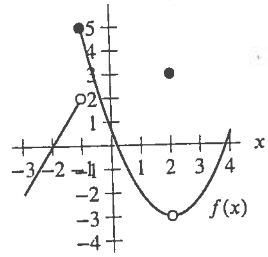

Figure 1.111

5.

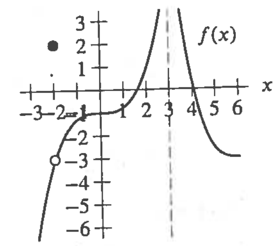

Figure 1.112

6. Assume $f(x)$ is continuous on an interval around $x = 0$, except possibly at $x = 0$. What does the table of values suggest as the value of $\lim_{x\to 0} f(x)$? Does the limit definitely have this value?

x	−0.1	−0.01	0.01	0.1
$f(x)$	1.987	1.999	1.999	1.987

7. Assume $g(t)$ is continuous on an interval around $t = 3$, except possibly at $t = 3$. What does the table of values suggest as the value of $\lim_{t\to 3} g(t)$? Does the limit definitely have this value?

t	2.9	2.99	3.01	3.1
$g(t)$	0.948	0.995	1.005	1.049

In Exercises 8–9,
(a) Make a table of values of $f(x)$ for $x = -0.1, -0.01, 0.01$, and 0.1.
(b) Use the table to estimate $\lim_{x\to 0} f(x)$.

8. $f(x) = \dfrac{\sin(5x)}{x}$

9. $f(x) = \dfrac{e^{3x} - 1}{x}$

10. Use a table of values to estimate $\lim_{x\to 1} (5 + \ln x)$.

In Exercises 11–16, is the function continuous on the interval?

11. $\dfrac{1}{x-2}$ on $[-1, 1]$

12. $\dfrac{1}{x-2}$ on $[0, 3]$

13. $\dfrac{1}{\sqrt{2x-5}}$ on $[3, 4]$

14. $2x + x^{-1}$ on $[-1, 1]$

15. $\dfrac{1}{\cos x}$ on $[0, \pi]$

16. $\dfrac{e^{\sin\theta}}{\cos\theta}$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$

17. Are the following functions continuous? Explain.

(a) $f(x) = \begin{cases} x & x \le 1 \\ x^2 & 1 < x \end{cases}$

(b) $g(x) = \begin{cases} x & x \le 3 \\ x^2 & 3 < x \end{cases}$

18. Let f be the function given by

$$f(x) = \begin{cases} 4 - x & 0 \le x \le 3 \\ x^2 - 8x + 17 & 3 < x < 5 \\ 12 - 2x & 5 \le x \le 6 \end{cases}.$$

(a) Find all values of x for which f is not continuous.
(b) List the largest open intervals on which f is continuous.

■ In Exercises 19–22, show there is a number c, with $0 \le c \le 1$, such that $f(c) = 0$.

19. $f(x) = x^3 + x^2 - 1$ **20.** $f(x) = e^x - 3x$

21. $f(x) = x - \cos x$ **22.** $f(x) = 2^x - 1/x$

■ In Exercises 23–28, use algebra to find the limit exactly.

23. $\lim_{x\to 2} \frac{x^2-4}{x-2}$ **24.** $\lim_{x\to -3} \frac{x^2-9}{x+3}$

25. $\lim_{x\to 1} \frac{x^2+4x-5}{x-1}$ **26.** $\lim_{x\to 1} \frac{x^2-4x+3}{x^2+3x-4}$

27. $\lim_{x\to 1} \frac{x^2+4}{x+8}$ **28.** $\lim_{h\to 0} \frac{(5+h)^2-5^2}{h}$

■ For Exercises 29–30, find the value of the constant k such that

29. $\lim_{x\to 5}(kx+10) = 20$ **30.** $\lim_{x\to 2} \frac{(x+6)(x-k)}{x^2+x} = 4$

■ In Exercises 31–34 find k so that the function is continuous on any interval.

31. $f(x) = \begin{cases} kx & x \le 3 \\ 5 & 3 < x \end{cases}$

32. $f(x) = \begin{cases} kx & 0 \le x < 2 \\ 3x^2 & 2 \le x \end{cases}$

33. $h(x) = \begin{cases} kx & 0 \le x < 1 \\ x+3 & 1 \le x \le 5. \end{cases}$

34. $g(t) = \begin{cases} t+k & t \le 5 \\ kt & 5 < t \end{cases}$

PROBLEMS

35. Which of the following are continuous functions of time?

(a) The quantity of gas in the tank of a car on a journey between New York and Boston.
(b) The number of students enrolled in a class during a semester.
(c) The age of the oldest person alive.

36. An electrical circuit switches instantaneously from a 6-volt battery to a 12-volt battery 7 seconds after being turned on. Graph the battery voltage against time. Give formulas for the function represented by your graph. What can you say about the continuity of this function?

37. A stone dropped from the top of a cliff falls freely for 5 seconds before it hits the ground. Figure 1.113 shows the speed $v = f(t)$ (in meters/sec) of the stone as a function of time t in seconds for $0 \le t \le 7$.

(a) Is f continuous? Explain your answer in the context of the problem.
(b) Sketch a graph of the height, $h = g(t)$, of the stone for $0 \le t \le 7$. Is g continuous? Explain.

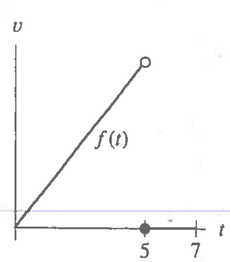

Figure 1.113

38. Beginning at time $t = 0$, a car undergoing a crash test accelerates for two seconds, maintains a constant speed for one second, and then crashes into a test barrier at $t = 3$ seconds.

(a) Sketch a possible graph of $v = f(t)$, the speed of the car (in meters/sec) after t seconds, on the interval $0 \le t \le 4$.
(b) Is the function f in part (a) continuous? Explain your answer in the context of this problem.

39. Discuss the continuity of the function g graphed in Figure 1.114 and defined as follows:

$$g(\theta) = \begin{cases} \frac{\sin\theta}{\theta} & \text{for } \theta \ne 0 \\ 1/2 & \text{for } \theta = 0. \end{cases}$$

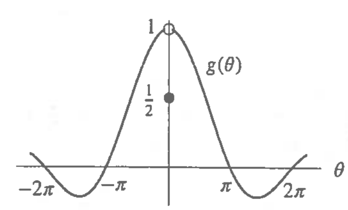

Figure 1.114

40. Is the following function continuous on $[-1, 1]$?

$$f(x) = \begin{cases} \frac{x}{|x|} & x \ne 0 \\ 0 & x = 0 \end{cases}$$

■ Estimate the limits in Problems 41–42 graphically.

41. $\lim_{x\to 0} \frac{|x|}{x}$

42. $\lim_{x\to 0} x \ln |x|$

■ In Problems 43–48, use a graph to estimate the limit. Use radians unless degrees are indicated by $\theta°$.

43. $\lim_{\theta\to 0} \frac{\sin(2\theta)}{\theta}$

44. $\lim_{\theta\to 0} \frac{\sin\theta°}{\theta°}$

45. $\lim_{h\to 0} \frac{e^h - 1}{h}$

46. $\lim_{h\to 0} \frac{e^{5h} - 1}{h}$

47. $\lim_{h\to 0} \frac{2^h - 1}{h}$

48. $\lim_{h\to 0} \frac{\cos(3h) - 1}{h}$

■ In Problems 49–54, find a value of k, if any, making $h(x)$ continuous on $[0, 5]$.

49. $h(x) = \begin{cases} k\cos x & 0 \le x \le \pi \\ 12 - x & \pi < x \end{cases}$

50. $h(x) = \begin{cases} kx & 0 \le x \le 1 \\ 2kx + 3 & 1 < x \le 5. \end{cases}$

51. $h(x) = \begin{cases} k\sin x & 0 \le x \le \pi \\ x + 4 & \pi < x \le 5. \end{cases}$

52. $h(x) = \begin{cases} e^{kx} & 0 \le x < 2 \\ x + 1 & 2 \le x \le 5. \end{cases}$

53. $h(x) = \begin{cases} 0.5x & 0 \le x < 1 \\ \sin(kx) & 1 \le x \le 5. \end{cases}$

54. $h(x) = \begin{cases} \ln(kx + 1) & 0 \le x \le 2 \\ x + 4 & 2 < x \le 5. \end{cases}$

55. **(a)** Use Figure 1.115 to decide at what points $f(x)$ is not continuous.
(b) At what points is the function $|f(x)|$ not continuous?

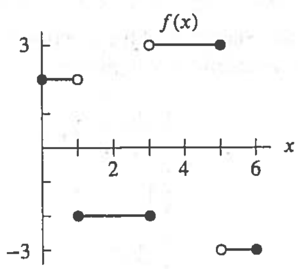

Figure 1.115

56. For t in months, a population, in thousands, is approximated by a continuous function

$$P(t) = \begin{cases} e^{kt} & 0 \le t \le 12 \\ 100 & t > 12. \end{cases}$$

(a) What is the initial value of the population?
(b) What must be the value of k?
(c) Describe in words how the population is changing.

57. A 0.6 ml dose of a drug is injected into a patient steadily for half a second. At the end of this time, the quantity, Q, of the drug in the body starts to decay exponentially at a continuous rate of 0.2% per second. Using formulas, express Q as a continuous function of time, t in seconds.

■ In Problems 58–61, at what values of x is the function not continuous? If possible, give a value for the function at each point of discontinuity so the function is continuous everywhere.

58. $f(x) = \frac{x^2 - 1}{x + 1}$

59. $g(x) = \frac{x^2 - 4x - 5}{x - 5}$

60. $f(z) = \frac{z^2 - 11z + 18}{2z - 18}$

61. $q(t) = \frac{-t^3 + 9t}{t^2 - 9}$

■ In Problems 62–65, is the function continuous for all x? If not, say where it is not continuous and explain in what way the definition of continuity is not satisfied.

62. $f(x) = 1/x$

63. $f(x) = \begin{cases} |x|/x & x \ne 0 \\ 0 & x = 0 \end{cases}$

64. $f(x) = \begin{cases} x/x & x \ne 0 \\ 1 & x = 0 \end{cases}$

65. $f(x) = \begin{cases} 2x/x & x \ne 0 \\ 3 & x = 0 \end{cases}$

66. Graph three different functions, continuous on $0 \le x \le 1$, and with the values in the table. The first function is to have exactly one zero in $[0, 1]$, the second is to have at least two zeros in the interval $[0.6, 0.8]$, and the third is to have at least two zeros in the interval $[0, 0.6]$.

x	0	0.2	0.4	0.6	0.8	1.0
$f(x)$	1.00	0.90	0.60	0.11	−0.58	−1.46

67. Let $p(x)$ be a cubic polynomial with $p(5) < 0$, $p(10) > 0$, and $p(12) < 0$. What can you say about the number and location of zeros of $p(x)$?

68. **(a)** What does a graph of $y = e^x$ and $y = 4 - x^2$ tell you about the solutions to the equation $e^x = 4 - x^2$?
(b) Evaluate $f(x) = e^x + x^2 - 4$ at $x = -4, -3, -2, -1, 0, 1, 2, 3, 4$. In which intervals do the solutions to $e^x = 4 - x^2$ lie?

69. **(a)** Does $f(x)$ satisfy the conditions for the Intermediate Value Theorem on $0 \le x \le 2$ if

$$f(x) = \begin{cases} e^x & 0 \le x \le 1 \\ 4 + (x - 1)^2 & 1 < x \le 2? \end{cases}$$

(b) What are $f(0)$ and $f(2)$? Can you find a value of k between $f(0)$ and $f(2)$ such that the equation $f(x) = k$ has no solution? If so, what is it?

70. Let $g(x)$ be continuous with $g(0) = 3$, $g(1) = 8$, $g(2) = 4$. Use the Intermediate Value Theorem to explain why $g(x)$ is not invertible.

71. By graphing $y = (1 + x)^{1/x}$, estimate $\lim_{x\to 0}(1 + x)^{1/x}$. You should recognize the answer you get. What does the limit appear to be?

72. Investigate $\lim_{h\to 0}(1 + h)^{1/h}$ numerically.

73. Let $f(x) = \sin(1/x)$.
(a) Find a sequence of x-values that approach 0 such that $\sin(1/x) = 0$.
[Hint: Use the fact that $\sin(\pi) = \sin(2\pi) = \sin(3\pi) = \ldots = \sin(n\pi) = 0$.]

Example 5 Use Theorems 1.3 and 1.4 to explain why the function is continuous.
(a) $f(x) = x^2 \cos x$ (b) $g(x) = \ln x$ (c) $h(x) = \sin(e^x)$

Solution
(a) Since $y = x^2$ and $y = \cos x$ are continuous everywhere, by Theorem 1.3 their product $f(x) = x^2 \cos x$ is continuous everywhere.
(b) Since $y = e^x$ is continuous everywhere it is defined and $\ln x$ is the inverse function of e^x, by Theorem 1.4, $g(x) = \ln x$ is continuous everywhere it is defined.
(c) Since $y = \sin x$ and $y = e^x$ are continuous everywhere, by Theorem 1.4 their composition, $h(x) = \sin(e^x)$, is continuous.

Exercises and Problems for Section 1.8 Online Resource: Additional Problems for Section 1.8

EXERCISES

1. Use Figure 1.121 to find the limits or explain why they don't exist.
(a) $\lim_{x\to-1^+} f(x)$ (b) $\lim_{x\to 0^-} f(x)$ (c) $\lim_{x\to 0} f(x)$
(d) $\lim_{x\to 1^-} f(x)$ (e) $\lim_{x\to 1} f(x)$ (f) $\lim_{x\to 2^-} f(x)$

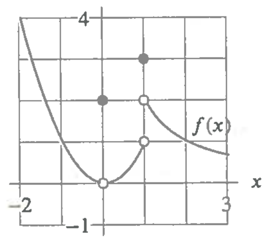

Figure 1.121

2. Use Figure 1.122 to estimate the following limits, if they exist.
(a) $\lim_{x\to 1^-} f(x)$ (b) $\lim_{x\to 1^+} f(x)$ (c) $\lim_{x\to 1} f(x)$
(d) $\lim_{x\to 2^-} f(x)$ (e) $\lim_{x\to 2^+} f(x)$ (f) $\lim_{x\to 2} f(x)$

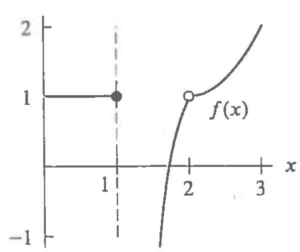

Figure 1.122

3. Use Figure 1.123 to find each of the following or explain why they don't exist.
(a) $f(-2)$ (b) $f(0)$ (c) $\lim_{x\to-4^+} f(x)$
(d) $\lim_{x\to-2^-} f(x)$ (e) $\lim_{x\to-2^+} f(x)$ (f) $\lim_{x\to 0} f(x)$
(g) $\lim_{x\to 2} f(x)$ (h) $\lim_{x\to 4^-} f(x)$

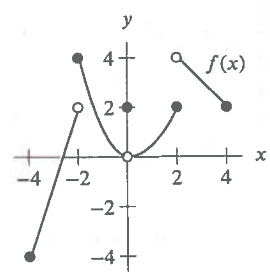

Figure 1.123

4. Use Figure 1.124 to find each of the following or explain why they don't exist.
(a) $f(0)$ (b) $f(4)$ (c) $\lim_{x\to-2^-} f(x)$
(d) $\lim_{x\to-2^+} f(x)$ (e) $\lim_{x\to-2} f(x)$ (f) $\lim_{x\to 0} f(x)$
(g) $\lim_{x\to 2^-} f(x)$ (h) $\lim_{x\to 2^+} f(x)$ (i) $\lim_{x\to 2} f(x)$
(j) $\lim_{x\to 4} f(x)$

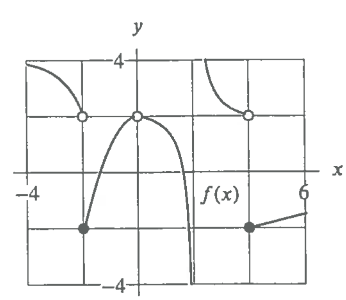

Figure 1.124

5. Use Figure 1.125 to estimate the following limits.
   (a) $\lim_{x\to\infty} f(x)$ (b) $\lim_{x\to-\infty} f(x)$

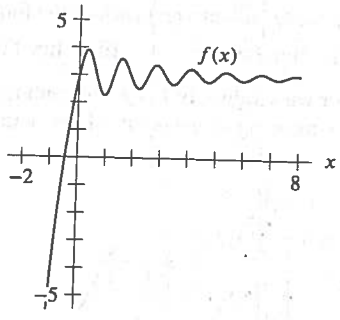

Figure 1.125

■ In Exercises 6–8, calculate the limit using the limit properties and $\lim_{x\to2} f(x) = 7$, $\lim_{x\to2} g(x) = -4$, $\lim_{x\to2} h(x) = \frac{1}{2}$.

6. $\lim_{x\to2}(f(x) - 2h(x))$

7. $\lim_{x\to2}(g(x))^2$

8. $\lim_{x\to2} \dfrac{f(x)}{g(x)\cdot h(x)}$

9. Using Figures 1.126 and 1.127, estimate
   (a) $\lim_{x\to1^-}(f(x)+g(x))$ (b) $\lim_{x\to1^+}(f(x)+2g(x))$
   (c) $\lim_{x\to1^-} f(x)g(x)$ (d) $\lim_{x\to1^+} \dfrac{f(x)}{g(x)}$

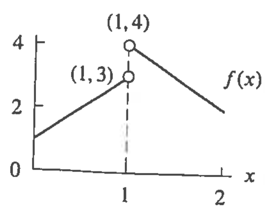

Figure 1.126

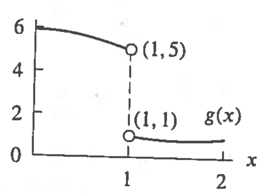

Figure 1.127

■ In Exercises 10–15, draw a possible graph of $f(x)$. Assume $f(x)$ is defined and continuous for all real x.

10. $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$

11. $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$

12. $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = \infty$

13. $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = 3$

14. $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-1} f(x) = 2$

15. $\lim_{x\to3} f(x) = 5$ and $\lim_{x\to-\infty} f(x) = \infty$

■ In Exercises 16–28, find the limits using your understanding of the end behavior of each function.

16. $\lim_{x\to\infty} x^2$ 17. $\lim_{x\to-\infty} x^2$ 18. $\lim_{x\to-\infty} x^3$

19. $\lim_{x\to\infty} x^3$ 20. $\lim_{x\to\infty} e^x$ 21. $\lim_{x\to\infty} e^{-x}$

22. $\lim_{x\to\infty} 5^{-x}$ 23. $\lim_{x\to\infty} \sqrt{x}$ 24. $\lim_{x\to\infty} \ln x$

25. $\lim_{x\to\infty} x^{-2}$ 26. $\lim_{x\to-\infty} x^{-2}$ 27. $\lim_{x\to-\infty} x^{-3}$

28. $\lim_{x\to\infty} \left(\frac{1}{2}\right)^x$

■ In Exercises 29–34, give $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} f(x)$.

29. $f(x) = -x^4$

30. $f(x) = 5 + 21x - 2x^3$

31. $f(x) = x^5 + 25x^4 - 37x^3 - 200x^2 + 48x + 10$

32. $f(x) = \dfrac{3x^3 + 6x^2 + 45}{5x^3 + 25x + 12}$

33. $f(x) = 8x^{-3}$

34. $f(x) = 25e^{0.08x}$

35. Does $f(x) = \dfrac{|x|}{x}$ have right or left limits at 0? Is $f(x)$ continuous?

■ In Exercises 36–38, use algebra to evaluate $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$, and $\lim_{x\to a} f(x)$ if they exist. Sketch a graph to confirm your answers.

36. $a = 4$, $f(x) = \dfrac{|x-4|}{x-4}$

37. $a = 2$, $f(x) = \dfrac{|x-2|}{x}$

38. $a = 3$, $f(x) = \begin{cases} x^2 - 2, & 0 < x < 3 \\ 2, & x = 3 \\ 2x + 1, & 3 < x \end{cases}$

PROBLEMS

39. By graphing $y = (1 + 1/x)^x$, estimate $\lim_{x\to\infty}(1 + 1/x)^x$. You should recognize the answer you get.

40. Investigate $\lim_{x\to\infty}(1 + 1/x)^x$ numerically.

41. (a) Sketch $f(x) = e^{1/(x^2+0.0001)}$ around $x = 0$.
    (b) Is $f(x)$ continuous at $x = 0$? Use properties of continuous functions to confirm your answer.

42. What does a calculator suggest about $\lim_{x\to 0^+} xe^{1/x}$? Does the limit appear to exist? Explain.

■ In Problems 43–52, evaluate $\lim_{x\to\infty}$ for the function.

43. $f(x) = \dfrac{x+3}{2-x}$

44. $f(x) = \dfrac{\pi + 3x}{\pi x - 3}$

45. $f(x) = \dfrac{x-5}{5+2x^2}$

46. $f(x) = \dfrac{x^2+2x-1}{3+3x^2}$

47. $f(x) = \dfrac{x^2+4}{x+3}$

48. $f(x) = \dfrac{2x^3-16x^2}{4x^2+3x^3}$

49. $f(x) = \dfrac{x^4+3x}{x^4+2x^5}$

50. $f(x) = \dfrac{3e^x+2}{2e^x+3}$

51. $f(x) = \dfrac{2^{-x}+5}{3^{-x}+7}$

52. $f(x) = \dfrac{2e^{-x}+3}{3e^{-x}+2}$

53. **(a)** Sketch the graph of a continuous function f with *all* of the following properties:
(i) $f(0) = 2$
(ii) $f(x)$ is decreasing for $0 \le x \le 3$
(iii) $f(x)$ is increasing for $3 < x \le 5$
(iv) $f(x)$ is decreasing for $x > 5$
(v) $f(x) \to 9$ as $x \to \infty$

(b) Is it possible that the graph of f is concave down for all $x > 6$? Explain.

54. Sketch the graph of a function f with *all* of the following properties:

(i) $f(0) = 2$ (ii) $f(4) = 2$
(iii) $\lim_{x\to-\infty} f(x) = 2$ (iv) $\lim_{x\to 0} f(x) = 0$
(v) $\lim_{x\to 2} f(x) = \infty$ (vi) $\lim_{x\to 4^-} f(x) = 2$
(vii) $\lim_{x\to 4^+} f(x) = -2$

55. Sketch the graph of a function f with *all* of the following properties:

(i) $f(-2) = 1$ (ii) $f(2) = -2$
(iii) $f(3) = 3$ (iv) $\lim_{x\to-\infty} f(x) = -2$
(v) $\lim_{x\to-1^-} f(x) = -\infty$ (vi) $\lim_{x\to-1^+} f(x) = \infty$
(vii) $\lim_{x\to 2} f(x) = 1$ (viii) $\lim_{x\to 3^-} f(x) = 3$
(ix) $\lim_{x\to 3^+} f(x) = 2$ (x) $\lim_{x\to\infty} f(x) = 1$

56. The graph of $f(x)$ has a horizontal asymptote at $y = -4$, a vertical asymptote at $x = 3$, and no other asymptotes.

(a) Find a value of a such that $\lim_{x\to a} f(x)$ does not exist.
(b) If $\lim_{x\to\infty} f(x)$ exists, what is its value?

57. A patient takes a 100 mg dose of a drug once daily for four days starting at time $t = 0$ (t in days). Figure 1.128 shows a graph of $Q = f(t)$, the amount of the drug in the patient's body, in mg, after t days.

(a) Estimate and interpret each of the following:
(i) $\lim_{t\to 1^-} f(t)$ (ii) $\lim_{t\to 1^+} f(t)$

(b) For what values of t is f not continuous? Explain the meaning of the points of discontinuity.

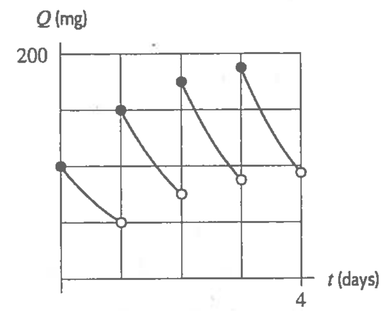

Figure 1.128

58. If $p(x)$ is the function on page 58 giving the price of renting a car, explain why $\lim_{x\to 1} p(x)$ does not exist.

59. Evaluate $\lim_{x\to 3} \dfrac{x^2+5x}{x+9}$ using the limit properties. State the property you use at each step.

60. Let $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. Give possible formulas for $f(x)$ and $g(x)$ if

(a) $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \infty$ **(b)** $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 3$
(c) $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$

61. **(a)** Rewrite $\dfrac{1}{x-5} - \dfrac{10}{x^2-25}$ in the form $f(x)/g(x)$ for polynomials $f(x)$ and $g(x)$.
(b) Evaluate the limit $\lim_{x\to 5}\left(\dfrac{1}{x-5} - \dfrac{10}{x^2-25}\right)$.
(c) Explain why you cannot use Property 4 of the limit properties to evaluate $\lim_{x\to 5}\left(\dfrac{1}{x-5} - \dfrac{10}{x^2-25}\right)$.

■ In Problems 62–63, modify the definition of limit on page 60 to give a definition of each of the following.

62. A right-hand limit

63. A left-hand limit

64. Use Theorem 1.2 on page 71 to explain why if f and g are continuous on an interval, then so are $f+g$, fg, and f/g (assuming $g(x) \neq 0$ on the interval).